Int. J. Solids Structures, 1971, Vol. 7, pp. 183 to 191. Pergamon Press. Printed in Great Britain

ON THE PARTIAL SIMULATION OF A NONCONSERVATIVE SYSTEM BY A CONSERVATIVE SYSTEM

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Abstract—The conditions under which a conservative system can reproduce the eigenvalues and eigenvectors of a related nonconservative system are found; these reinforce conclusions arrived at intuitively and from physical reasoning. Modal frequencies for a nonconservative follower force system are then obtained from an associated conservative system, the exercise being done both numerically and experimentally. The difficulties of simulating more complicated systems are discussed.

1. INTRODUCTION

THE nonconservative system considered in the present work is one in which the non-conservative forces acting are "circulatory" to use Ziegler's nomenclature [1]. Dissipative, gyroscopic and instationary forces are assumed not to be present. Typical circulatory forces are these generated fluid-dynamically; the lift force on an oscillating wing or the thrust force from a rocket or jet engine are examples. But the forces in this group may be sub-divided further, for in the examples just mentioned there is a fundamental difference. In the former case, the circulatory force has a varying magnitude whilst in the latter it may be assumed sensibly constant. Although both types lead to the same kind of stiffness matrix in the equations of motion, it is only the constant magnitude force which is now considered.

Sometimes, the creation of a force like this in the laboratory presents awkward practical problems, and the simulation of this force wherever possible by a conservative force would be very convenient. However, because of the differing nature of the fundamental properties of the conservative and nonconservative systems, the simulation could only work in a situation where the two systems behave in similar ways; that is, when the nonconservative system is not operating in a regime of oscillatory instability. (The conservative system cannot become dynamically unstable since, by definition, it has no energy source from which to supply the extra kinetic energy involved in the instability.) The restriction to the sub-critical load range implies a partial simulation only, but it is later seen that there is another sense in which the simulation is a partial one, for only one eigenmode can be reproduced at a time. The manner by which a partial simulation is achieved and the restrictions it is subject to are now discussed.

2. CONDITIONS THAT AN EIGENVECTOR AND EIGENVALUE OF A NONCONSERVATIVE SYSTEM BE IDENTICAL WITH AN EIGENVECTOR AND EIGENVALUE OF A CONSERVATIVE SYSTEM

We need first of all to derive the generalized forces arising from the application of a constant magnitude external force P to any elastic body having arbitrary generalized coordinates of displacement from equilibrium $q_1, q_2, \ldots, q_n$. The force P is assumed to be either conservative or nonconservative-circulatory so that in each case it is dependent entirely on $q_1, q_2, \ldots, q_n$ [1]. The force and body system is assumed scleronomic and linear. The force has, at any instant direction cosines $\tilde{l}_k$ ($k = 1, 2, 3$) which may be written using Taylor's series

$$\tilde{l}_k = l_k + \left(\frac{\partial l_k}{\partial q_1}\right) q_1 + \left(\frac{\partial l_k}{\partial q_2}\right) q_2 + \ldots + \frac{1}{2}\left(\frac{\partial^2 l_k}{\partial q_1^2}\right) q_1^2 + \frac{1}{2}\left(\frac{\partial^2 l_k}{\partial q_2^2}\right) q_2^2 + \frac{\partial^2 l_k}{\partial q_1 \partial q_2} q_1 q_2 + \ldots$$

$$= l_k + \mathbf{q}^T \mathbf{l}'_k + \frac{1}{2}\mathbf{q}^T \mathbf{l}''_k \mathbf{q} + \ldots \tag{1}$$

where the higher order terms are not shown. The terms on the right hand side are referred to equilibrium conditions, the dashes indicate that the matrix coefficients are partial differentials with respect to $q_1, q_2, \ldots$, and superscript T indicates a transposed matrix. Similarly, the displacement components $\tilde{x}_k$ of the point of application may be written

$$\tilde{x}_k = x_k + \left(\frac{\partial x_k}{\partial q_1}\right) q_1 + \left(\frac{\partial x_k}{\partial q_2}\right) q_2 + \ldots + \frac{1}{2}\left(\frac{\partial^2 x_k}{\partial q_1^2}\right) q_1^2 + \frac{1}{2}\left(\frac{\partial^2 x_k}{\partial q_2^2}\right) q_2^2 + \left(\frac{\partial^2 x_k}{\partial q_1 \partial q_2}\right) q_1 q_2 + \ldots$$

$$= x_k + \mathbf{q}^T \mathbf{x}'_k + \frac{1}{2}\mathbf{q}^T \mathbf{x}''_k \mathbf{q} + \ldots . \tag{2}$$

A set of virtual displacements $\delta\tilde{x}_k$ is therefore

$$\delta\tilde{x}_k = \boldsymbol{\delta}\mathbf{q}^T \mathbf{x}'_k + \boldsymbol{\delta}\mathbf{q}^T \mathbf{x}''_k \mathbf{q} + \ldots \tag{3}$$

and the work done by force P is:

$$\delta W = P \sum_{k=1}^{3} \tilde{l}_k \delta\tilde{x}_k$$

$$= P \sum_{k=1}^{3} (l_k \boldsymbol{\delta}\mathbf{q}^T \mathbf{x}'_k + l_k \boldsymbol{\delta}\mathbf{q}^T \mathbf{x}''_k \mathbf{q} + \boldsymbol{\delta}\mathbf{q}^T \mathbf{x}'_k \mathbf{l}'^T_k \mathbf{q} + \ldots) \tag{4}$$

from (1) and (3).

If P is the only external force acting then from the definition of equilibrium the first term becomes zero and the generalized forces are now

$$\frac{\partial \mathbf{w}}{\partial \mathbf{q}} = P \sum_{k=1}^{3} (l_k \mathbf{x}''_k \mathbf{q} + \mathbf{x}'_k \mathbf{l}'^T_k \mathbf{q}) \tag{5}$$

where the higher order terms are neglected. The presence of P therefore introduces additional stiffness terms into the equations of motion which may be written

$$P\mathbf{c}\mathbf{q} = P \sum_{k=1}^{3} (l_k \mathbf{x}''_k + \mathbf{x}'_k \mathbf{l}'^T_k)\mathbf{q}. \tag{6}$$

The matrix $\mathbf{c}$ is symmetric if P is conservative and asymmetric if P is nonconservative [1]. For a number of forces $P_1, P_2, \ldots, P_m$ acting on the body simultaneously the additional terms are simply; $\sum_{j=1}^{m} P_j \mathbf{c}_j$ so that the equations of motion of the system are

$$\mathbf{a}\ddot{\mathbf{q}} + \left[\mathbf{b} + \sum_{j=1}^{m} P_j \mathbf{c}_j\right]\mathbf{q} = \mathbf{0} \tag{7}$$

where $\mathbf{a}$ is an inertia matrix and $\mathbf{b}$ a stiffness matrix. We now let these equations apply to the nonconservative case, and, in addition, we let *all* the forces P_j be circulatory. For the conservative system the same elastic body is considered to be acted on by conservative forces which have the same (constant) magnitude as $P_1, P_2, \ldots, P_m$ and which act at the same points on the body. The equations of motion in this case are

$$\mathbf{a}\ddot{\mathbf{q}}+\left[\mathbf{b}+\sum_{j=1}^{m} P_j\mathbf{c}_j^*\right]\mathbf{q} = \mathbf{0} \tag{8}$$

where the same generalized coordinates are used. The matrices $\mathbf{c}_j^*$ are symmetric whilst $\mathbf{c}_j$ are asymmetric. A particular solution to (7) is $\mathbf{q} = \mathbf{Q}e^{\lambda t}$, and to (8) is $\mathbf{q} = \mathbf{Q}^*e^{\lambda^* t}$, where λ and λ^* may be complex and t represents time, so that

$$\left[\mathbf{a}\lambda^2+\mathbf{b}+\sum_{j=1}^{m} P_j\mathbf{c}_j\right]\mathbf{Q} = \mathbf{0} \tag{9}$$

$$\left[\mathbf{a}\lambda^{*2}+\mathbf{b}+\sum_{j=1}^{m} P_j\mathbf{c}_j^*\right]\mathbf{Q}^* = \mathbf{0}. \tag{10}$$

If the ith eigenvector of each system is to be the same, i.e.

$$\mathbf{Q} = \mathbf{Q}^* = \mathbf{r}_i,$$

then

$$\left[\mathbf{a}(\lambda_i^2-\lambda_i^{*2})+\sum_{j=1}^{m} P_j(\mathbf{c}_j-\mathbf{c}_j^*)\right]\mathbf{r}_i = \mathbf{0}. \tag{11}$$

In addition, if the ith eigenvalues are to be equal, then

$$\sum_{j=1}^{m} P_j(\mathbf{c}_j-\mathbf{c}_j^*)\mathbf{r}_i = \mathbf{0}. \tag{12}$$

The equating of λ_i^2 and λ_i^{*2} in (11) implies that λ_i and λ_i^* are either both real, imaginary or complex. The last case is not tenable here because it in turn implies oscillatory instability, and the conservative system is not able to become unstable in this way [2]. Hence our attention is restricted to comparing the two systems only when the nonconservative system is *not* operating in a regime of oscillatory instability. The P_j's in (12) are not in general subject to such a set of linear constraints so that for (12) to be satisfied in all circumstances

$$(\mathbf{c}_j-\mathbf{c}_j^*)\mathbf{r}_i = \mathbf{0} \tag{13}$$

or

$$\sum_{k=1}^{3} (l_{kj}\mathbf{x}_{kj}''+\mathbf{x}_{kj}'\mathbf{l}_{kj}'^T-l_{kj}^*\mathbf{x}_{kj}''^*-\mathbf{x}_{kj}'^*\mathbf{l}_{kj}'^{T*})\mathbf{r}_i = \mathbf{0} \tag{14}$$

from (6). But $\mathbf{x}_{kj}' = \mathbf{x}_{kj}'^*$ and $\mathbf{x}_{kj}'' = \mathbf{x}_{kj}''^*$ since the conservative and nonconservative forces have the same points of application, and if the two sets of forces possess the same orientation at equilibrium, then $l_{kj} = l_{kj}^*$ in addition. Thus, (14) becomes

$$\sum_{k=1}^{3} \mathbf{x}_{kj}'(\mathbf{l}_{kj}'-\mathbf{l}_{kj}'^*)^T\mathbf{r}_i = \mathbf{0}. \tag{15}$$

The matrix pre-multiplying $\mathbf{r}_i$ is of unit rank [since $\mathbf{x}'_{kj}$ is a column and $(\mathbf{l}'_{kj}-\mathbf{l}'^{*}_{kj})$ a row matrix], so only one physical condition is represented by (15). This is

$$\mathbf{l}'_{kj} = \mathbf{l}'^{*}_{kj} \qquad k = 1, 2, 3 \tag{16}$$

which states that the rates of change of the direction cosines of the jth conservative force with respect to the generalized coordinates must equal those of the jth nonconservative force. Since the choice of generalized coordinates is arbitrary, the situation is achieved only if the two sets of forces maintain the same direction cosines, or alignment in space, when the elastic body undergoes the same deformation in each case.

3. BECK'S NONCONSERVATIVE SYSTEM AND ITS ASSOCIATED CONSERVATIVE SYSTEM

The problem of the stability of a cantilever column acted on by a follower force at its free end [see Fig. 1(i)] was initially examined by Beck [3]. The follower force is such that it remains tangential to the beam at its point of application for all possible configurations of the beam; such a force can be shown by a simple demonstration to be nonconservative (see Bolotin [2]). To obtain a conservative system that will reproduce a non-complex eigenvalue and eigenvector of Beck's system it is necessary to use an identical cantilever with an end force of the same magnitude that remains tangential to the cantilever tip in the current eigenmode. A system which satisfies the requirements is shown in Fig. 1(ii); the line of action of the end force passes through a point lying on the line of the undeflected cantilever, and the distance d of the point to the tip can be adjusted to make the load tangential. Willems [4] used this arrangement and simulated a single mode of Beck's system for a range of loads, but above a certain value of load, the simulation broke down; the situation was later analysed by Huang, *et al.* [5], although no suggestion was given as to how Willems' system might be used to obtain a full and correct simulation in the stable

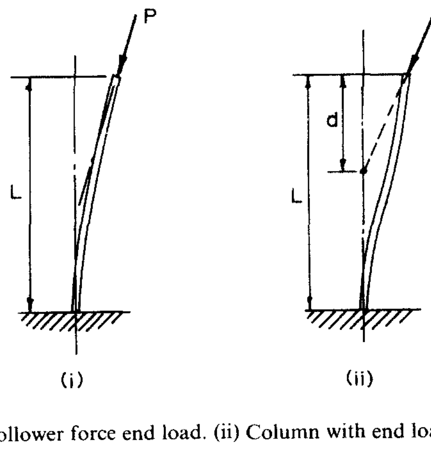

FIG. 1. (i) Column with follower force end load. (ii) Column with end load through a fixed point.

regime. The two systems will now be examined together as an illustration of the general principles outlined in Section 2.

The elements of the matrices appearing in (7) and (8) can be shown to be

$$a_{ij} = m \int_0^L f_i(x) f_j(x)\,\mathrm{d}x \tag{17}$$

$$b_{ij} = EI \int_0^L f_i''(x) f_j''(x)\,\mathrm{d}x \tag{18}$$

$$c_{ij} = f_i(L) f_j'(L) - \int_0^L f_i'(x) f_j'(x)\,\mathrm{d}x \tag{19}$$

$$c_{ij}^{*} = f_i(L) f_j(L)/d - \int_0^L f_i'(x) f_j'(x)\,\mathrm{d}x \tag{20}$$

where m is the mass per unit length and EI the flexural rigidity of the cantilever, L is its length and d is the distance from the free end of the point of intersection of the force vector with the beam axis. The arbitrary mode shape $f_i(x)$ is that associated with the generalized coordinate q_i and differentiation with respect to x is indicated by a dash. Only one external constant magnitude force P is involved.

Application of the condition (13) to (19) and (20) leads to the requirement that in the ith eigenmode

$$d = \frac{\mathbf{f}(L)\mathbf{r}_i}{\mathbf{f}'(L)\mathbf{r}_i} \tag{21}$$

where $\mathbf{f}(L)$ is a row matrix of the tip amplitudes $f_j(L)$ in each arbitrary mode and $\mathbf{f}'(L)$ is a row matrix of the tip slopes. Equation (21) states, in effect, that d is equal to the tip amplitude divided by the tip slope in the ith eigenmode i.e. that the force vector is tangential at that point.

4. NUMERICAL CHECK ON EIGENVALUES

Beck's system and its associated conservative system described in the preceding Section were used as the basis for a numerical demonstration of the simulation. The data used applied to the experimental model described in Section 5 and illustrated in Fig. 3.

The eigenvalues for the conservative system were computed for the case when the force was tangential to the cantilever tip, and then compared with the (non-complex) eigenvalues for the associated Beck's system. The values of modal frequency against load are plotted for each case in Fig. 2 and it may be seen that the two sets of points lie on the same curve.

The method of obtaining the curve in the conservative case was first of all to obtain the plots of natural frequency against load for a number of different values of d. Some of these are shown by the dashed curves in Fig. 2. The slope at the cantilever tip was computed at various stations along the curves, and the interpolated points where this coincided with the tangent of the load vector were used to draw the full curve. In the nonconservative case the frequencies were computed directly. At loads higher than that for frequency coalescence (point C in Fig. 2) the nonconservative system becomes unstable; the imaginary (frequency) part of the eigenvalues is not shown.

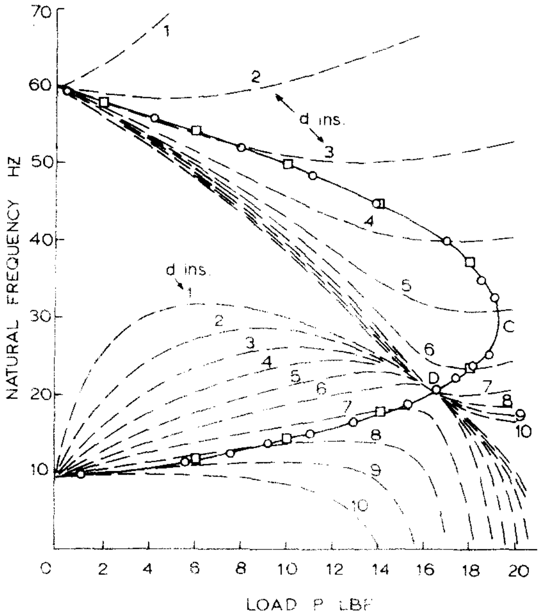

FIG. 2. Numerical results. —□— Modal frequencies on Beck's system. ---- Natural frequencies of conservative system. ○ Points obtained from conservative system, at which force vector is tangential to tip of cantilever.

For the sake of simplicity, the number of degrees of freedom was restricted to two; the arbitrary modes used were the two lower frequency normal modes of the unloaded beam. This restriction is justified by the small size of the errors incurred (see Appendix).

5. EXPERIMENTAL CHECK

The experimental counterpart of the numerical check described in Section 4 was carried out using the model shown in Fig. 3. The load is applied by two equally tensioned wires which pass through adjustable columns on each side of the cantilever. The beam can be vibrated by an electromagnetic exciter attached near the fixed end.

The two lower natural frequencies were found at each load condition and value of d by performing a simple resonance test, and the variation of the two lower natural frequencies with load is shown by the dashed curves in Fig. 4. The configurations for which the tip slope coincided with the load vector are indicated by points on these curves. Each configuration was found by either a visual method in which a stroboflash was used or by an optical method in which a beam of light was reflected from a mirror fixed to the cantilever tip and the angular deflection of the beam measured. The full line curve drawn through the points therefore represents the experimentally obtained variation of modal frequency with load (up to the critical load for oscillatory instability) for the nonconservative Beck's system.

6. DISCUSSION

It has been shown that a conservative system can simulate a particular type of non-conservative system provided that (i) the latter is not operating in a regime of oscillatory instability, and that (ii) the motion is confined to one eigenvector. The first proviso is obvious from physical considerations but to illustrate the second consider the contrary situation;

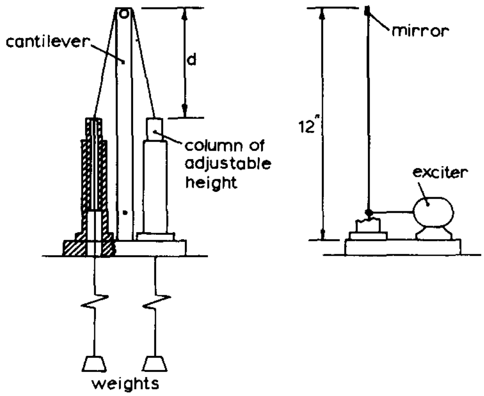

FIG. 3. Experimental model.

in this case, assuming it were not unstable, the Beck system if given an impulse would vibrate in some combination of its eigenmodes. The conservative system if similarly subjected would vibrate in a like manner, but since the natural frequencies and mode shapes would generally not be the same as those of the former (except in the case of the simulating and simulated mode) the overall motion would not be reproduced. For example in Fig. 2 at $P = 9$ lbf the value of d for simulation of the lower mode needs to be about 8 in. The upper mode frequency at $P = 9$ lbf is 51 Hz for the nonconservative system (42 Hz for the conservative system) and to attain this value d needs to be about 3 in. Simulation of the two modes simultaneously could therefore not be achieved. In general, for a system having

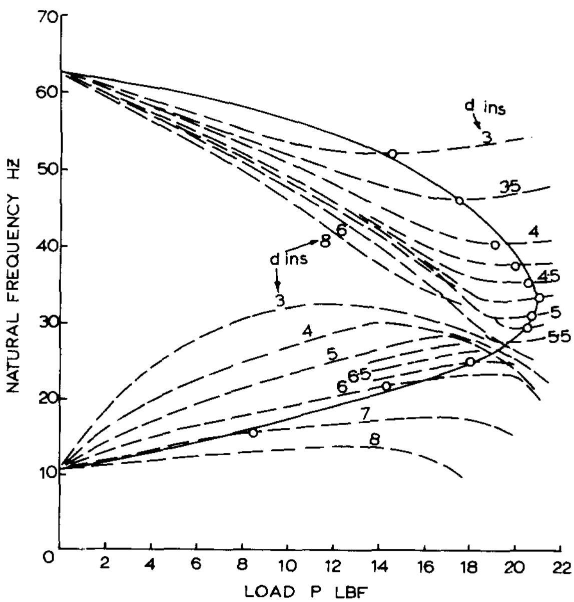

FIG. 4. Experimental results. —— Natural frequencies of conservative system. —○— Modal frequencies of Beck's system obtained from conservative system.

n degrees of freedom, simulation of each mode in the subcritical regime would require a set of n uniquely different conservative systems; however, we have seen that this set can be obtained from the differing geometrical configurations of a single *mechanical* system.

A limitation which is entirely practical is one which was nearly reached in the present case. Inspection of Fig. 2 shows for large loads that it is the upper mode of each system comprising the family of conservative systems which simulates both the upper and lower modes of the Beck system. The lower mode of the former, however, shows for the values of d involved (around 5 and 6 in.) a strong tendency to become statically unstable as the load increases. The point of static instability or divergence for a particular configuration is marked by the frequency becoming zero, and it may be seen that this occurs at a load only just greater than the load for which the upper mode simulates a mode of the Beck system. Although the behaviour of the lower mode is of no interest in the present context, were the divergence to occur at a load lower than that for simulation, over a large enough range of configurations, then the laboratory simulation, restricted in the manner already discussed, could not be achieved at all over that range.

Other nonconservative systems in the circulatory category may be examined, of course; one which would appear to hold promise is the follower force system examined by Herrmann and Bungay [6] in which the applied force moves through an angle which is some proportion of the angle moved through by the local structure under deformation (in Beck's case the proportion is 1 : 1). Forces of variable magnitude, such as the aerodynamic lift on an oscillating aerofoil, are not covered in the present theory, but even here a simulation might be possible; in the example quoted the lift force could be replaced by a component of a constant magnitude force that changes direction as the angle of attack of the wing changes.

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APPENDIX

A comparison of important values of non-dimensional frequency and load parameters is given in the Table below. In column 1 are the values obtained using the present two-

TABLE 1

	1	2
$mL^4\omega_c^2/\Pi^4$	1·2708	1·2457
P_CL^2/Π^4EI	2·0338	2·0316
P_DL^2/Π^4EI	1·7225	1·626

degree of freedom Rayleigh–Ritz approach. In column 2 are the exact values taken from Ref. [5]. The values of frequency parameter at the natural frequencies of the unloaded cantilever are identical in each case. Suffix C refers to the point of frequency coalescence of the nonconservative system; ω_c is the circular frequency there. Suffix D refers to the point of frequency coalescence of the conservative system (it occurs for a particular value of $d = 0{\cdot}542L$). These points are shown on Fig. 2.

(*Received* 15 *January* 1970; *revised* 13 *April* 1970)

Абстракт—Даются условия, при которых консервативная система может воспроизводить собственные значения и собственные векторы смежной неконсервативной системы. Эти усилительные выводы приходят интуитивно и на основе физических рассуждений. Получаются затем модальные частоты для неконсервативной системы следящей силы, выходя из соответствующей консервативной системы. Пример дается так численно, как и экспериментально. Обсуждаются трудности более сложных моделированных систем.